Global stabilization of stochastic nonlinear systems

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Abstract

We consider stochastic nonaffine nonlinear control systems $dx_t = f(x_t,u)\,dt + g(x_t,u)\,d\omega_t$, ω being a standard Wiener process, for which we give a sufficient condition for global stabilization by a bounded smooth state feedback which is explicitely given. This condition generalizes the well known Jurdjevic-Quinn result for deterministic affine control systems.

1 Introduction

In this paper, we investigate the global stabilization problem for stochastic nonlinear control differential equations (written in the sens of Itô) of the form

$$x_t = x_0 + \int_0^t f(x_s,u)\,ds + \sum_{j=1}^{p} \int_0^t g_j(x_s,u)\,d\omega_s^j \qquad (1)$$

where $u = (u^1,\dots,u^m)^{\mathrm{T}}$ is a bounded R^m–valued control law, $f, g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are bounded smooth ($\mathcal{C}^\infty$) functions with $f(0,0) = g_j(0,0) = 0,\ 1 \le j \le p$ and $\{\omega_t,\ t \ge 0\}$ is a standard $\mathbb{R}^p$–valued Wiener process defined on an usual probability space $(\Omega, \mathcal{F}, P)$. System (1) is said globally stabilizable if there exists a feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$ with $u(0) = 0$ such that the zero solution of the closed-loop system

$$x_t = x_0 + \int_0^t f(x_s,u(x_s))\,ds + \sum_{j=1}^{p} \int_0^t g_j(x_s,u(x_s))\,d\omega_s^j$$

is globally asymptotically stable in probability (G.A.S.P.) (see [3]). Our goal is to give a sufficient condition for global stabilization of (1). This condition generalizes the well known Jurdjevic-Quinn [2] result for deterministic affine control systems (see [4] and references therein)

$$\dot{x} = X(x) + \sum_{i=1}^{m} u^i Y^i(x) \qquad (2)$$

In [4], it is proved that if there exists a positive definite and proper smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that:
(i) the Lie-derivative of V with respect to X satisfies $XV(x) \le 0, \forall x \in \mathbb{R}^n$; (ii) the set $\{x \in \mathbb{R}^n | X^{k+1}V(x) = X^k Y^i(V) = 0,\ k \in \mathbb{N},\ i = 1,\dots,m\}$ is reduced to $\{0\}$; then system (2) is globally stabilizable by means of the feedback law $u^i(x) = -Y^i V(x)$. [1] extends the Jurdjevic-Quinn theorem to the particular class of stochastic control affine nonlinear systems

$$x_t = x_0 + \int_0^t [X_0(x_s) + \sum_{i=1}^{m} u^i Y_0^i(x_s)]ds + \sum_{j=1}^{p} \int_0^t X_j(x_s) dz_s^j \qquad (3)$$

for which the coefficients associated with the noise do not depend on the control. For (3) the associated infinitesimal generator $\mathcal{L}$ satisfies $\mathcal{L}V(x) = LV(x) + \sum_{i=1}^m u^i Y_0^i V(x)$, where L is the second order differential operator: $L = X_0 + \frac{1}{2}\sum_{j=1}^p X_j^2$, and V is a Lyapunov function. So, the feedback $u^i(x) = -Y_0^i V(x)$ yields $\mathcal{L}V(x) \le 0$ if $LV(x) \le 0$, which allows to establish in [1], under stochastic analogous conditions to (i) and (ii), that this feedback globally stabilizes (3). Contrarily to (3), for (1) where the random parametric excitation depends on the control, the linearity on u disapears in $\mathcal{L}V(x)$, and so, it is not obvious to prove the existence of a state feedback law $u(x)$ yielding $\mathcal{L}V(x) \le 0$ if $LV(x) \le 0$.

2 Main result

In order to state our result, we need the following notations. Let X_0, Y_0^i, X_j and Y_j^i, $1 \le j \le p$, $1 \le i \le m$, be the vector fields defined on $\mathbb{R}^n$ by $X_0(x) = f(x,0)$, $Y_0^i(x) = (\partial f/\partial u^i)(x,0)$, $X_j(x) = g_j(x,0)$ and $Y_j^i(x) = (\partial g_j/\partial u^i)(x,0)$. Moreover, denote by L and L_i, $1 \le i \le m$, the second order differential operators associated with (1) defined for any function Ψ in $\mathcal{C}^2(\mathbb{R}^n)$ by

$$\begin{aligned} L\Psi(x) &= \langle X_0(x), \nabla\Psi(x)\rangle \\ &\quad + \tfrac{1}{2}\sum_{j=1}^{p} \mathrm{Tr}\left(X_j(x)X_j^{\mathrm{T}}(x)(\partial^2\Psi/\partial x^2)(x)\right) \qquad (4) \end{aligned}$$

$$\begin{aligned} L_i\Psi(x) &= \langle Y_0^i(x), \nabla\Psi(x)\rangle + \tfrac{1}{2}\sum_{j=1}^{p} \mathrm{Tr}\left([X_j(x){Y_j^i}^{\mathrm{T}}(x) \right. \\ &\quad \left. + Y_j^i(x)X_j^{\mathrm{T}}(x)](\partial^2\Psi/\partial x^2)(x)\right) \qquad (5) \end{aligned}$$

Besides, for $x \in \mathbb{R}^n$ and $v, w \in \mathbb{R}^m$ we set

$$\tilde{f}(x,v,w) = \int_0^1 (1-t)(\partial^2 f/\partial u^2)(x,tv)w^2 dt$$

$$\tilde{g}_j(x,v,w) \quad = \quad \int_0^1 (1-t)(\partial^2 g_j/\partial u^2)(x,tv)w^2 dt$$

where
$w^2 = (w,w)$ and $(\partial^2 f/\partial u^2)(x,tv)$, $(\partial^2 g_j/\partial u^2)(x,tv) \in \mathcal{L}_2(\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^n)$ (the space of bilinear applications from $\mathbb{R}^m\times\mathbb{R}^m$ to $\mathbb{R}^n$) are respectivelly the second order derivatives of f and g_j with respect to u at (x,tv), and we consider the $n\times n$ matrices

$$\begin{aligned} A_j(x,v,w) \quad &= \quad X_j(x)\tilde{g}_j^{\mathrm{T}}(x,v,w) + \tilde{g}_j(x,v,w)X_j^{\mathrm{T}}(x) \\ &+ \sum_{i=1}^m v^i \left[Y_j^i(x)\tilde{g}_j^{\mathrm{T}}(x,v,w) + \tilde{g}_j(x,v,w){Y_j^i}^{\mathrm{T}}(x)\right] \\ &+ \sum_{i_1,i_2=1}^m w^{i_1}w^{i_2}Y_j^{i_1}(x){Y_j^{i_2}}^{\mathrm{T}}(x) + \tilde{g}_j(x,v,v)\tilde{g}_j^{\mathrm{T}}(x,v,w) \end{aligned}$$

Finally, we assume that there exists $V(x)$ proper positive definite and smooth such that
(h1) $LV(x) \le 0,\ \forall x\in\mathbb{R}^n$;
(h2) The set $W = \{x\in\mathbb{R}^n \mid L^{k+1}V(x) = L^kL_iV(x) = 0,\ k\in\mathbb{N},\ i=1,\ldots,m\}$ is reduced to $\{0\}$;
and we set

$$\begin{aligned} \varphi(x,v,w) \quad &= \quad \langle \tilde{f}(x,v,w), \nabla V(x)\rangle \\ &+ \frac{1}{2}\sum_{j=1}^p \mathrm{Tr}\left(A_j(x,v,w)(\partial^2V/\partial x^2)(x)\right) \end{aligned} \tag{6}$$

Notice that the real valued function φ is homegeneous of degree 2 with respect to w. Using the recursive notation $L^{k+1}\Psi(x) = LL^k\Psi(x)$, $L^0\Psi(x) = \Psi(x)$, for $k\in\mathbb{N}$ and $\Psi\in\mathcal{C}^\infty(\mathbb{R}^n)$, our main result can now be stated as follows.

Theorem 1 *If the conditions* **(h1)** *and* **(h2)** *hold, then for any $\eta>0$ and any smooth functions $K_1(x)$ and $K_2(x)$ satisfying, $\forall x\in\mathbb{R}^n$, $K_1(x)+K_2(x)\neq 0$ and*

$$K_1(x) \quad \ge \quad \sup_{\|v\|\le\eta,\|w\|=1} |\varphi(x,v,w)| \tag{7}$$

$$K_2(x) \quad \ge \quad \left\|\left(L_1V(x),..,L_mV(x)\right)\right\| \tag{8}$$

the stochastic control system (1) is globally stabilizable by means of the feedback law

$$u(x) = \frac{-\eta}{\eta K_1(x)+K_2(x)}\left(L_1V(x),..,L_mV(x)\right)^{\mathrm{T}} \tag{9}$$

which satisfies $\|u(x)\|\le\eta,\ \forall x\in\mathbb{R}^n$.

Proof: The inequality $\|u(x)\|\le\eta$ is immediate. Moreover, the closed-loop system is given by

$$\begin{aligned} x_t \quad &= \quad x_0 + \int_0^t \Big[X_0(x_s) + \sum_{i=1}^p u^i(x_s)Y_0^i(x_s) \\ &+ \tilde{f}(x_s,u(x_s),u(x_s))\Big]ds + \sum_{j=1}^p \int_0^t \Big[X_j(x_s) \\ &+ \sum_{i=1}^p u^i(x_s)Y_j^i(x_s) + \tilde{g}_j(x_s,u(x_s),u(x_s))\Big]d\omega_s^j \end{aligned} \tag{10}$$

The infinitesimal generator $\mathcal{L}$ associated with (10) satisfies

$$\begin{aligned} \mathcal{L}V(x) \quad &= \quad \Big\langle X_0(x_s) + \sum_{i=1}^p u^i(x_s)Y_0^i(x_s) \\ &+ \tilde{f}\big(x_s,u(x_s),u(x_s)\big), \nabla V(x)\Big\rangle + \frac{1}{2}\sum_{j=1}^p \mathrm{Tr}\Big(\big[X_j(x_s) \\ &+ \sum_{i=1}^m u^i(x_s)Y_j^i(x_s) + \tilde{g}_j\big(x_s,u(x_s),u(x_s)\big)\big]\big[X_j(x_s) \\ &+ \sum_{i=1}^m u^i(x_s)Y_j^i(x_s) + \tilde{g}_j\big(x_s,u(x_s),u(x_s)\big)\big]^{\mathrm{T}}\frac{\partial^2V}{\partial x^2}(x)\Big) \end{aligned}$$

and by a simple computation one gets from (4), (5) and (6) $\mathcal{L}V(x) = LV(x) + \sum_{i=1}^m u^i(x)L_iV(x) + \varphi\big(x,u(x),u(x)\big)$. It follows that $\mathcal{L}V(x) = LV(x)$ if $u(x)=0$ and otherwise, from (9) and the homogeneity property of $\varphi(x,v,w)$ with respect to w one gets

$$\mathcal{L}V(x) = LV(x) - \frac{1}{K(x)}\|u(x)\|^2\Big[1 - K(x)\varphi\big(x,u,\frac{u}{\|u\|}\big)\Big]$$

where $K(x) = \eta/(\eta K_1(x) + K_2(x))$. Besides, by (7), (8) and $\|u(x)\| \le \eta$ one has $1 - K(x)\,\varphi\,(x,u(x),u(x)/\|u(x)\|) \ge 0$ and so one gets from **(h1)** $\mathcal{L}V(x)\le 0,\ \forall x\in\mathbb{R}^n$. Besides, according to the stochastic version of LaSalle's invariance principle (see [3]), the stochastic process x_t converges in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}V(x_t)=0$ for all $t\ge 0$. Therefore, in order to prove that the zero solution of (10) is G.A.S.P. it must be shown that for any complete solution x_t of (10) along which $\mathcal{L}V(x_t)=0$ for all $t\ge 0$, one has necessarily $x_t\equiv 0$. Notice that, from (7), (8) and (9), one has if $u(x)\neq 0$ then $K_2(x)\neq 0$ and so $1 - K(x)\varphi(x,u(x),u(x)/\|u(x)\|)\neq 0$ and, from (9), it turns out that $\mathcal{L}V(x)=0$ if and only if $LV(x)=0$ and $L_iV(x)=0,\ i=1,\ldots,m$. So, for any complete solution x_t of (10) for which $\mathcal{L}V(x_t)=0$ for all $t\ge 0$, successive differentiations by means of Itô's formula yield $L^{k+1}V(x_t) = L^kL_iV(x_t) = 0$, for $t\ge 0$, $k\in\mathbb{N}$, and $i=1,\ldots,m$. Hence, by assumption **(h2)**, it follows that $x_t\equiv 0$ which completes the proof.

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